Forward Time Central Space and Saulyev’s Method for Solving one Dimensional Parabolic Equations with Non Local Condition

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Abstract:

The purpose of this study is to test and compare two explicit numerical approaches, the conditionally stable forward time central space FTCS and unconditionally stable Saulyev’s method, for solving parabolic equations with a nonlocal boundary condition using different step sizes of time (t). These methods are based on finite different schemes. All computations are carried out using mathematica wolfram 8.0 software.

We have employed these numerical schemes to show the accuracy of their solutions and since they are explicit, their results are compared. The stability of these numerical schemes is also discussed.

It should be noted that FTCS method is closer to the exact solution so long as $r \leq 0.5$ in comparison with Saulyev’s method. On contrary the Saulyev’s method is more stable and gives a good approximation in case $r > 0.5$. .

Keywords: Finite difference method; Saulyev’s method; nonlocal boundary condition; Stability;

الملخص:

الهدف من هذه الدراسة إختبار ومقارنة طريقتان من الطرق العددية الصريحة وهي طريقة الفروق الأمامية للزمن والمركزية للمسافة FTCS والتي تكون مستقرة بشروط وطريقة ساولوفي والتي تكون مستقرة بدون شروط لحل المعادلات الجزئية المكافئة مع الشرط غير المحلي باستعمال أحجام مختلفة من الخطوات للزمن (t). هذه الطرق مبنية على طريقة

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الفروق المنتهية. كل العمليات الحسابية تم تنفيذها باستخدام برنامج الماتيماتيكا وولفرام 8.0.

تم دراسة هذا النوع من الطرق العددية لمعرفة الدقة في الحلول، ولكونهما طرقاً صريحة تمت المقارنة بين نتائجهما ومناقشة استقرارهما.

من خلال النتائج يمكننا التوصل إلى أن طريقة ال FTCS تكون قريبة في نتائجها من الحل النظري بشرط $r \leq 0.5$ بالمقارنة مع طريقة ساولوف. في المقابل نجد أن طريقة ساولوف مستقرة بدون شروط وتعطي نتائج جيدة في حالة $r > 0.5$.

الكلمات المفتاحية: طريقة الفروق المنتهية، طريقة ساولوف، الشروط الحدية غير المحلية، الإستقرار.

1. Introduction

Many physical phenomena are formulated by nonclassical parabolic initial-boundary value problems in one space variable which involve an integral term over the spatial domain of a function of the desired solution. This integral term may appear in a boundary condition, in which case the boundary condition is called nonlocal (Dehghan, 2005).

1.1 Parabolic equation with nonlocal boundary condition

Let consider one form of this equation which in the following

$$\frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t) + s(x,t), \quad 0 < x < L, x > 0. \qquad (1)$$

Subject to the initial condition

$$u(x,0) = f(x), \quad 0 < x < L, \qquad (2)$$

the boundary condition

$$u(l,t) = g(t), \quad 0 < t \leq T, \qquad (3)$$

and the nonlocal boundary condition

$$\int_0^b u(x,t)\,dx = m(t), \quad 0 < t \leq T, \quad 0 < b < 1, \qquad (4)$$

where f, g, b, s and are known, while the function u is to be determined (Dehghan,2001).

2. Schemes for the parabolic equation with nonlocal boundary condition.

In this section we obtain some finite difference approximation schemes that can solve parabolic equations of the kind (1-4).

To develop a finite difference method we need to introduce grid points, let N and M be positive integers, and $h = \Delta x,\ k = \Delta t,\ h = \frac{L}{N}$, $k = \frac{T}{M}$. Define the partition points $x_i = i\,h$, $i = 0,1,\ldots N$, $t_j = jk$, $j = 0,1,\ldots M$.

A point of the form (x_i, t_j) is called a grid point and we are interested in computing approximate solution values at the grid points (Han, 2005).

The notation u_i^j is use for an approximation to $u_i^j \equiv u(x_i, t_j)$.

2.1 Forward Time Central Space (FTCS) method.

Forward time and central space can be obtained by replacing the time derivative with a forward difference and the second spatial derivative with a central difference and negligent the truncation errors, we get

$$\left.\frac{\partial u}{\partial t}\right|_i^j = \frac{u_i^{j+1} - u_i^j}{k} \quad \text{and} \quad \left.\frac{\partial^2 u}{\partial x^2}\right|_i^j = \frac{u_i^{j+1} - 2u_i^j + u_{i-1}^j}{h^2}. \tag{5}$$

Substituting the above equations into (1), then the forward time central space scheme can be written as

$$\frac{u_i^{j+1} - u_i^j}{k} = \frac{u_i^{j+1} - 2u_i^j + u_{i-1}^j}{h^2} + s_i^j, \tag{6}$$

or

$$u_i^{j+1} = r\left(u_{i+1}^j + u_{i-1}^j\right) + (1 - 2r)u_i^j + ks_i^j, \tag{7}$$

where $r = \frac{k}{h^2}$. Equation (7) expresses u_i^{j+1} explicitly in terms of u_i^j. The method is thus an explicit method and the unknown u_i^{j+1} can be directly determined if u_i^j is known (Yau, 1994).

2.2 Saulyev's method

To quote from (Jie Sun, 2008), the first derivative in time and the second derivative in space of the function value can be written as

$$\frac{\partial u_i^j}{\partial t} = \frac{u_i^{j+1} - u_i^j}{k} + o(k), \tag{8}$$

$$\frac{\partial^2 u_i^j}{\partial x^2} = \frac{1}{h}\left(\frac{\partial u_{i+(1/2)}^j}{\partial x} - \frac{\partial u_{i-(1/2)}^j}{\partial x}\right) + o(h^2). \tag{9}$$

Applying

$$\frac{\partial u_{i+(1/2)}^j}{\partial x} = \frac{\partial u_{i+(1/2)}^{j+1}}{\partial x} - k\frac{\partial^2 u(i+(1/2), t_j + \theta k)}{\partial x \partial t}, \quad 0 \le \theta \le 1. \tag{10}$$

Applying (8-10) in equation (1), we get

$$\frac{u_i^{j+1} - u_i^j}{k} = \frac{\alpha}{h}\left(\frac{\partial u_{i+(1/2)}^{j+1}}{\partial x} - \frac{\partial u_{i-(1/2)}^j}{\partial x}\right) + \frac{1-\alpha}{h}\left(\frac{\partial u_{i+(1/2)}^j}{\partial x} - \frac{\partial u_{i-(1/2)}^j}{\partial x}\right) + s_i^j$$
$$- \alpha\frac{k}{h}\frac{\partial^2 u(i+(1/2), t_j + \theta k)}{\partial x \partial t} + o(k + h^2), \quad 0 \le \alpha \le 1.$$

Hence, the above equation can be written in the following form

$$\frac{u_i^{j+1} - u_i^j}{k} = \frac{\alpha}{h^2}\left(u_{i+1}^{j+1} - u_i^{j+1} - u_i^j + u_{i-1}^j\right)$$
$$+ \frac{1-\alpha}{h^2}\left(u_{i+1}^j - u_i^j - u_i^j + u_{i-1}^j\right) + s_i^j. \tag{11}$$

Similarly we can obtain

$$\frac{u_i^{j+1} - u_i^j}{k} = \frac{\alpha}{h^2}\left(u_{i+1}^j - u_i^j - u_i^{j+1} + u_{i-1}^{j+1}\right)$$
$$+ \frac{1-\alpha}{h^2}\left(u_{i+1}^j - u_i^j - u_i^j + u_{i-1}^j\right) + s_i^j. \tag{12}$$

Putting =1 in equation (11) and (12), we obtain Saulyev's scheme

$$(1 + r)u_i^{j+1} - ru_{i-1}^{j+1} = (1-r)u_i^j + ru_{i+1}^j + ks_i^j, \tag{13}$$

$$(1 + r)u_i^{j+1} - ru_{i+1}^{j+1} = (1-r)u_i^j + ru_{i-1}^j + ks_i^j, \tag{14}$$

where $r = \frac{k}{h^2}$.

2.2.1 Saulyev's first kind formula

According to Dehghan (2005), Saulyev's scheme has two kinds: first and second kind formula. The explicit formula can be obtained from equation (13) in the following form

$$u_i^{j+1} = \frac{1}{(1+r)}[ru_{i-1}^{j+1} + (1-r)u_i^j + ru_{i+1}^j + ks_i^j]. \quad (15)$$

For $i = 1,2, \ldots N-2, N-1$.

This formula is used in case of that u_i^{j+1} is known and u_N^{j+1} is unknown. This means that we first put $i = 1$ and then $i = 2, \ldots$ and finally $i = N-1$ (Dehghan, 2005).

2.2.2 Saulyev's second kind formula

The explicit Saulyev's second kind formula can be obtained from equation (14) as

$$u_i^{j+1} = \frac{1}{(1+r)}[ru_{i+1}^{j+1} + (1-r)u_i^j + ru_{i-1}^j + ks_i^j. \quad (16)$$

For $i = N-1, N-2, \ldots, 2, 1$.

This formula is used in case of that u_i^{j+1} is unknown and u_N^{j+1} is known. This means that we first put $i = N-1$ and then $i = N-2, \ldots$ and finally $i = 1$ (Dehghan, 2005).

3. Treatment of the nonlocal condition of the parabolic equation.

We consider the nonlocal condition from equation (4) by Simpson's one-third rule; we use this formula due to a higher order of its truncation error. According to Rao (2006), Simpson's one-third rule formula can be written as

$$\int_a^b u(x)\,dx \approx \frac{h}{3}[u(x_0) + 4u(x_1) + \ldots + 2u(x_{N-2}) + 4u(x_{N-1}) + u(x_N)]$$

$$\approx \frac{h}{3}\left[x_0 + 4\sum_{i=1}^{\frac{N}{2}} u(x_{2i-1}) + 2\sum_{i=1}^{\frac{N}{2}-1} u(x_{2i}) + u(x_N)\right], \quad (17)$$

where $x_i = ih$, $h = \frac{b-a}{N}$, $i = 0,1, \ldots N$.

The error committed in Simpson's one-third rule is given by

$$E \approx \frac{-Nh^5}{180} f^{iv}(\xi) = -\frac{(b-a)^5}{180N^4} f^{iv}(\xi), \tag{18}$$

where $a = x_0 < \xi < x_n = b$ (for N subintervals of length h).
Applying Simpson's one-third rule formula in equation (4) we obtain

$$m^{j+1} \approx \frac{h}{3}[u_0^{j+1} + 4u_1^{j+1} + 2u_2^{j+1} + h + u_N^{j+1}]. \tag{19}$$

Hence

$$m^{j+1} \approx \frac{h}{3}\left[u_0^{j+1} + 4\sum_{i=1}^{\frac{N}{2}} u_{2i-1}^{j+1} + 2\sum_{i=1}^{\frac{N}{2}-1} u_{2i}^{j+1} + u_N^{j+1}\right]. \tag{20}$$

4. Consistency, Stability & Convergence

In order for approximate solutions to converge to true solutions of PDE as step sizes in time and space go to zero, two conditions must be met:

_Consistency:- which means that local truncation error goes to zero as step sizes go to zero (i.e. discrete problem approximates right continuous problem).

_stability:- which essentially means that approximate solution remains bounded.

In general, a numerical procedure is unstable if errors introduced into the calculations grow at an exponential rate as the computation proceeds (Boyce, 1977).

Lax equivalence theorem says that consistency and stability are together necessary and sufficient for convergence. Neither condition a lone is sufficient to guarantee convergence.

In the forward method with $o(k + h^2)$ of convergence, the stability will occur only if $0 \le r \le 0.5$ Since $r = \frac{k}{h^2}$, this inequality requires that h^2 and k must be chosen so that $\frac{k}{h^2} \le 0.5$.

Hence we call the forward - difference method conditional stable and remark that the method converges to the solution of the parabolic equation. This severe restriction on time step makes explicit method relatively inefficient compared to Saulyev's method, although it has the same order of convergence.

Dehghan (2005) pointed out that the Saulyev's procedures are very simple to implement and this method is explicit as well as unconditionally stable, although the standard fully explicit schemes have restriction on stability and are only useful over small time steps.

5. Numerical Experiments

In this section, we present some numerical results to confirm our theoretical analysis.

Example 1. Let us consider the equations (1-4) with

$f(x) = e^x, \quad g(t) = \frac{e}{1+t^2}, \quad b = 0.6,$

$m(t) = \frac{e^{0.6}-1}{1+t^2}, \quad s(x,t) = \frac{-(1+t)^2 e^x}{(1+t^2)^2}, \quad x \in (0,1)$ and $t \in (0,T]$,

which is easily seen to have the exact solution (Dehghan ,2005):

$$u(x,t) = \frac{e^x}{1+t^2}. \qquad (21)$$

We perform two explicit numerical schemes and compare the exact solution of the equation above with different step sizes of time t.

Simpson's rule is used to get, $u(0,t)$, $t \in (0,T]$ from the nonlocal condition at each method.

5.1 Forward Time Central Space Method

We have applied FTCS method to solve the equation (1) for $x \in (0,1)$, and Simpson's rule for solving $u(0,t)$, $t \in (0,T]$.

The numerical result of FTCS method are obtained and compared with the exact solution at $h = 0.1$, $k = 0.005$ and $t = 0.005$ as shown in Table (1) and Figure (1).

Table (1): results of FTCS Scheme for with $h = 0.1$, $k = 0.005$ and $t = 0.005$.

Value of	FTCS Scheme	Exact Solution	Relative Error
0.0	0.99995	.99998	2.0706×10^{-5}
0.1	1.10512	1.10517	2.0706×10^{-5}
0.2	1.22135	1.22141	2.0706×10^{-5}
0.3	1.3498	1.34986	2.0706×10^{-5}

Value of	FTCS Scheme	Exact Solution	Relative Error
0.4	1.49176	1.49182	2.0706×10^{-5}
0.5	1.64865	1.64872	2.0706×10^{-5}
0.6	1.82204	1.82212	2.0706×10^{-5}
0.7	2.01366	2.01375	2.0706×10^{-5}
0.8	2.22544	2.22554	2.0706×10^{-5}
0.9	2.45949	2.4596	2.0706×10^{-5}
1.0	2.71821	2.71821	0000

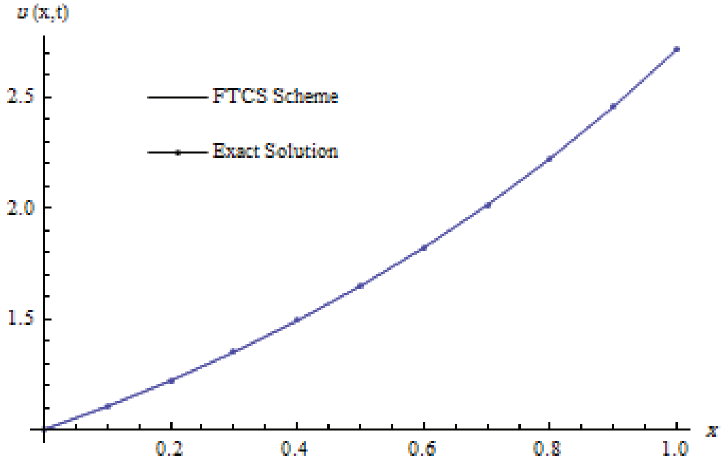

Figure (1): FTCS method and exact solution at $h = 0.1$, $k = 0.005$ and $t = 0.005$.

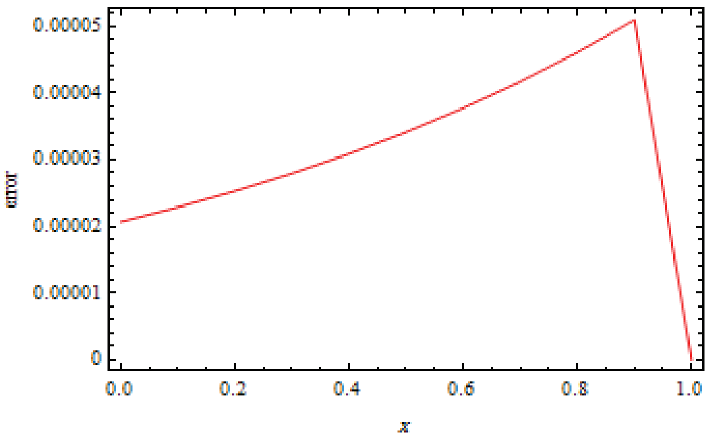

Figure (2): Error curve of FTCS method $h = 0.1$, $k = 0.005$ and $t = 0.005$.

Table (2) and Figure (3) illustrate the comparison between the result of FTCS method and the exact solution at $h = 0.1$, $k = 0.005$ and $t = 0.01$.

Table (2): results of FTCS Scheme for $u(x,t)$ with $h = 0.1$, $k = 0.005$ and $t = 0.01$.

Value of	FTCS Scheme	Exact Solution	Relative Error
0.0	0.99987	0.9999	3.1395×10^{-5}
0.1	1.10526	1.10506	1.8360×10^{-4}
0.2	1.22123	1.22128	4.1259×10^{-5}
0.3	1.34967	1.34972	4.1259×10^{-5}
0.4	1.49161	1.49168	4.1259×10^{-5}
0.5	1.64849	1.64856	4.1259×10^{-5}
0.6	1.82186	1.82194	4.1259×10^{-5}
0.7	2.01347	2.01355	4.1259×10^{-5}
0.8	2.22523	2.22532	4.1259×10^{-5}
0.9	2.45928	2.45936	2.9817×10^{-5}
1.0	2.71801	2.71801	0.0000

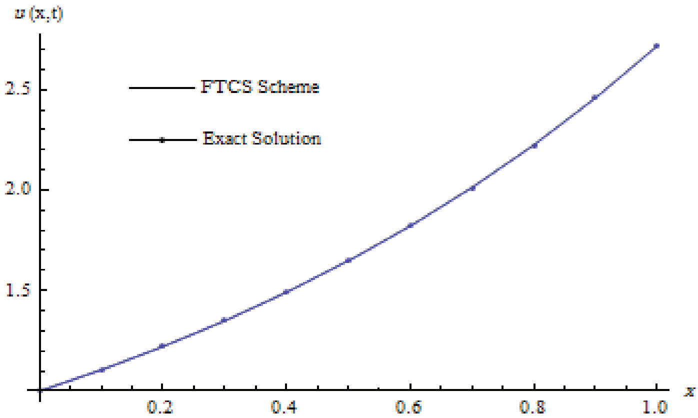

Figure (3): FTCS method and exact solution at $h = 0.1$, $k = 0.005$ and $t = 0.01$.

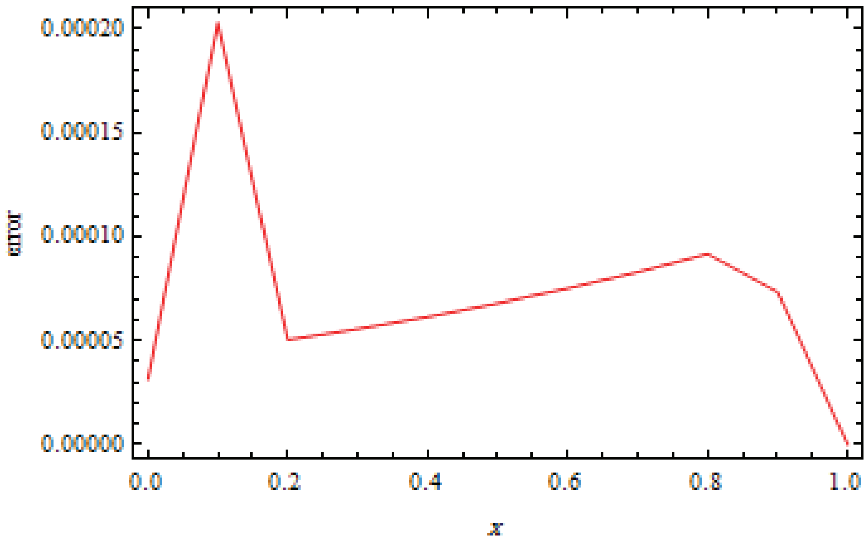

Figure (4): Error curve of FTCS method at $h = 0.1$, $k = 0.005$ and $t = 0.01$.

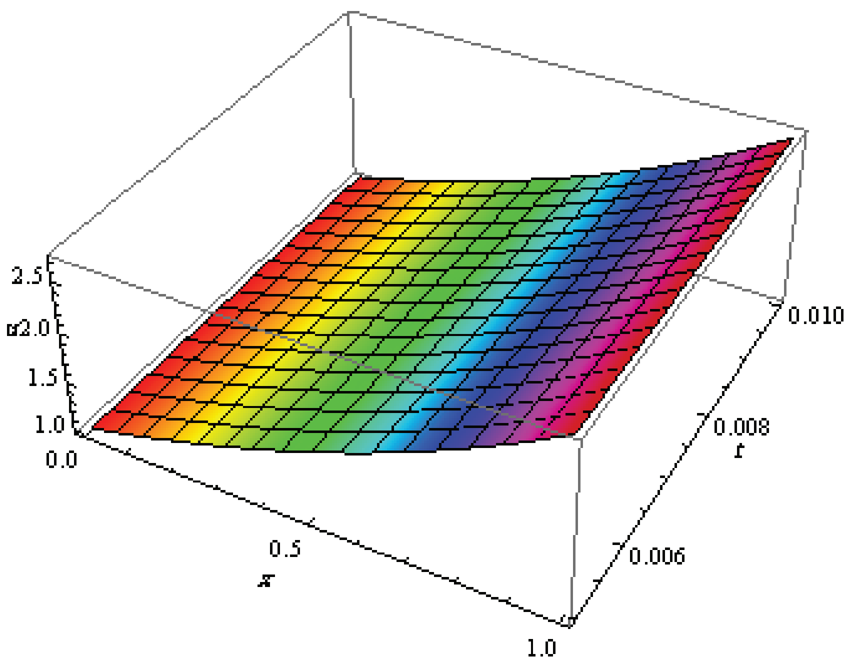

Figure (5): solutions of FTCS method in three dimension.

Table (3) and Figure (6) are shown the comparison between the result of FTCS method and the exact solution at $h = 0.1$, $k = 0.02$ and $t = 0.08$.

Table (3): results of FTCS Scheme for $u(x,t)$ with $h = 0.1$, $k = 0.02$ and $t = 0.08$.

Value of	FTCS Scheme	Exact Solution	Relative Error
0.0	-5.63275	0.993641	6.6688
0.1	2.90981	1.09814	1.64975
0.2	0.767782	1.21364	3.6737×10^{-1}
0.3	1.4132	1.34127	5.3624×10^{-2}
0.4	1.48021	1.48234	1.4367×10^{-3}
0.5	1.63588	1.63824	1.4367×10^{-3}
0.6	1.80793	1.81053	1.4367×10^{-3}
0.7	2.00622	2.00095	2.6372×10^{-3}
0.8	2.19179	2.21139	8.8616×10^{-3}
0.9	2.46082	2.44396	6.8975×10^{-3}
1.0	2.701	2.701	0.0000

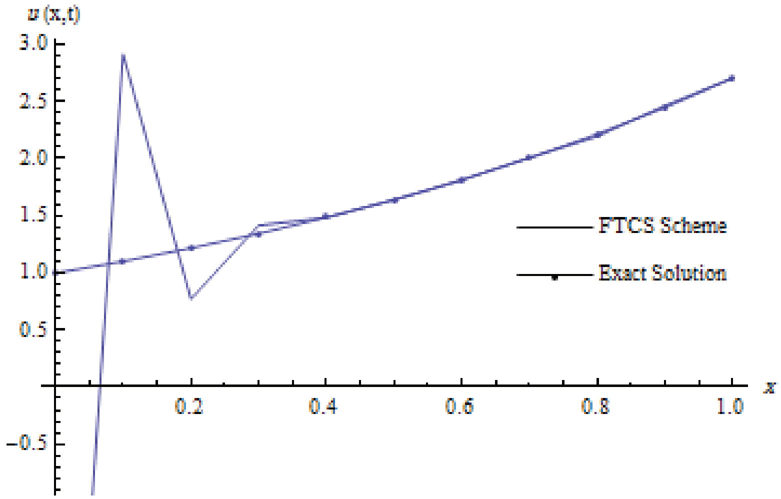

Figure (6): FTCS method and exact solution $h = 0.1$, $k = 0.02$ and $t = 0.08$.

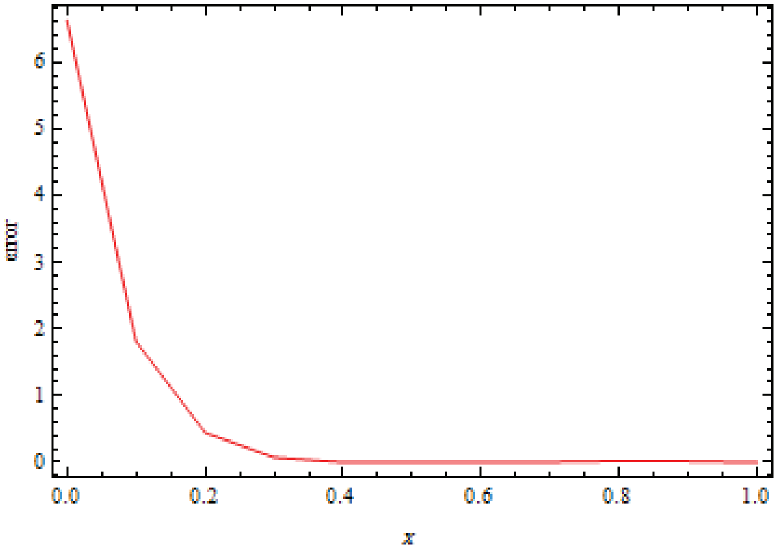

Figure (7): Error curve of FTCS method at $h = 0.1$, $k = 0.02$ and $t = 0.08$.

As we can see from the results, forward time central space gave a good approximation of equation (1) in each different step size which is noticed from the relative errors in the Tables (1), (2) and Figures (1)-(5).

According to the solution in Table (3), Figure (6) and Figure (7) the unstable behaviour is made apparent because the value of $r = 2$ for this simulation. An r value that is significantly higher than the stability limit of 0.5.

5.2 Saulyev's Method

In this section we perform the Saulyev's second kind formula to solve the equation (1) for x h $(0,1)$, and Simpson's rule to get $(0,t)$, t h $(0,T]$. Table (4) and Figure (8) are shown the comparison between the result of Saulyev's method and the exact solution at $h = 0.1$, $k = 0.005$ and $t = 0.005$.

Table (4): results of Saulyev's Scheme for $u(x,t)$ with $h = 0.1$, $k = 0.005$ and $t = 0.005$.

Value of	Saulyev's Scheme	Exact Solution	Relative Error
0.0	0.99995	0.99998	2.3241×10^{-5}
0.1	1.10512	1.10517	2.3240×10^{-5}

Value of	Saulyev's Scheme	Exact Solution	Relative Error
0.2	1.22134	1.22141	2.3235×10^{-5}
0.3	1.34979	1.34986	2.3221×10^{-5}
0.4	1.49175	1.49182	2.3184×10^{-5}
0.5	1.64864	1.64872	2.3085×10^{-5}
0.6	1.82203	1.82212	2.2814×10^{-5}
0.7	2.01366	2.01375	2.2808×10^{-5}
0.8	2.22544	2.22554	2.0088×10^{-5}
0.9	2.45951	2.4596	1.4680×10^{-5}
1.0	2.71821	2.71821	0.0000

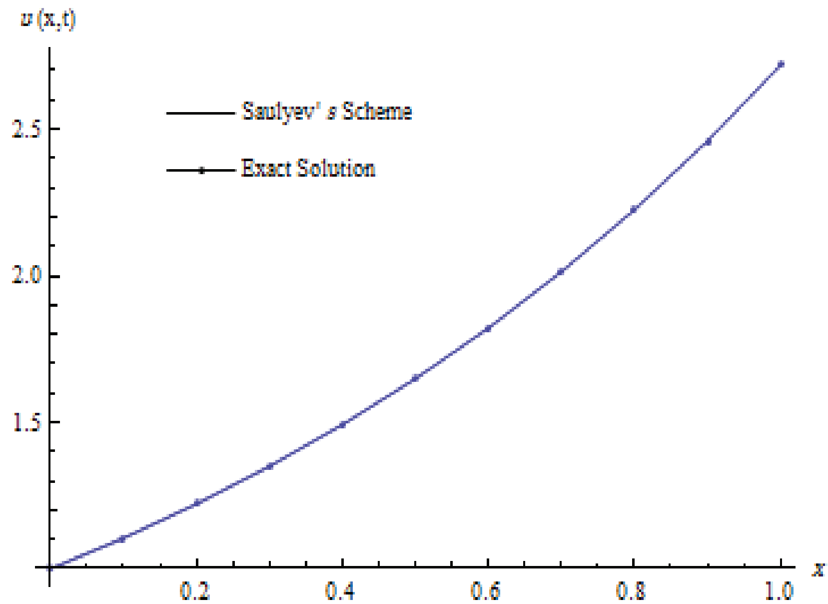

Figure (8): Saulyev's method and exact solution at $h = 0.1$, $k = 0.005$ and $t = 0.005$.

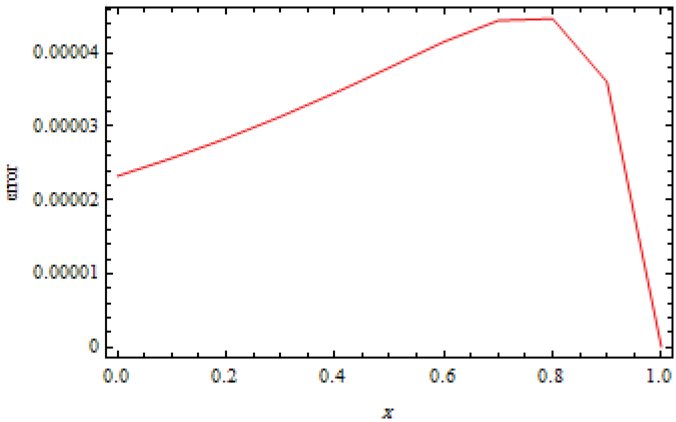

Figure (9): Error curve of Saulyev's method at $h = 0.1$, $k = 0.005$ and $t = 0.005$.

Table (5) and Figure (10) represent the comparison between the solution of Saulyev's method and the exact solution at $h = 0.1$, $k = 0.005$, and $t = 0.01$. Further, the three dimensional solution is illustrated in Figure (12) at $h = 0.1$, $k = 0.005$ and $t = 0.01$.

Table (5): results of Saulyev's Scheme for $u(x,t)$ with $h = 0.1$, $k = 0.005$ and $t = 0.01$.

Value of	Saulyev's Scheme	Exact Solution	Relative Error
0.0	1.0003	0.9999	3.9607×10^{-4}
0.1	1.10519	1.10506	1.1904×10^{-4}
0.2	1.22122	1.22128	4.9069×10^{-5}
0.3	1.34966	1.34972	4.9001×10^{-5}
0.4	1.4916	1.49168	4.8836×10^{-5}
0.5	1.64848	1.64856	4.8430×10^{-5}
0.6	1.82185	1.82194	4.7448×10^{-5}
0.7	2.01346	2.01355	4.5108×10^{-5}
0.8	2.22523	2.22532	3.9644×10^{-5}
0.9	2.45929	2.45936	2.7215×10^{-5}
1.0	2.71801	2.71801	0.0000

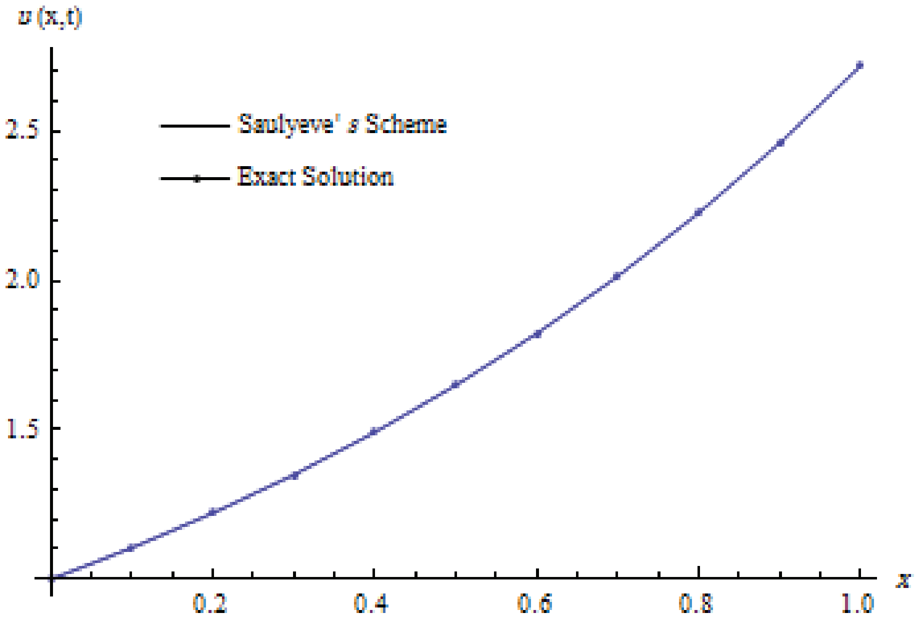

Figure (10): Saulyev's method and exact solution $h = 0.1$, $k = 0.005$ and $t = 0.01$.

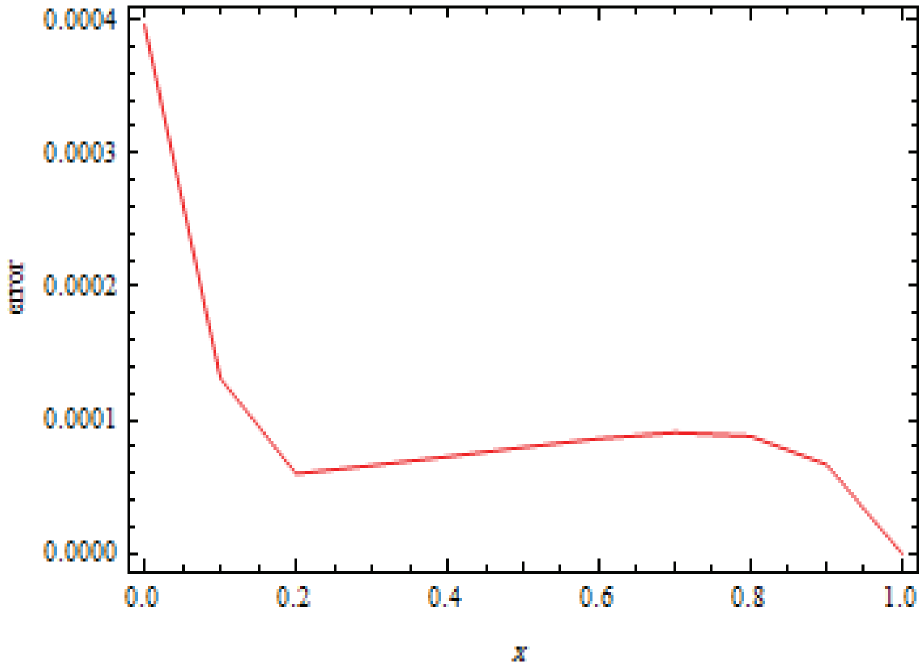

Figure (11): Error curve of Saulyev's method at $h = 0.1$, $k = 0.005$ and $t = 0.01$.

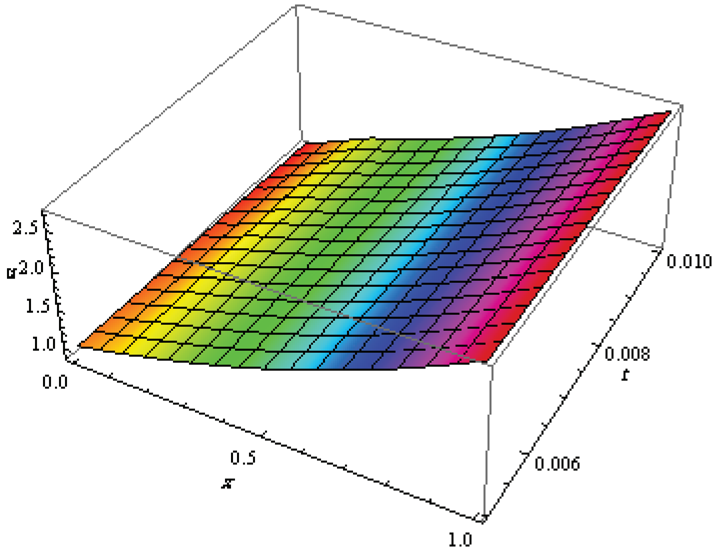

Figure (12): solutions of Saulyev's method in three dimension.

The comparison between the result of Saulyev's method and the exact solution are displayed in Table (6) and Figure (13) at $h = 0.1$, $k = 0.02$, and $t = 0.08$.The error curves are plotted in Figures (9), (11) and (14) at different step sizes.

Table (6): results of Saulyev's Scheme for $u(x,t)$ with $h = 0.1$, $k = 0.02$ and $t = 0.08$.

Value of	Saulyev's Scheme	Exact Solution	Relative Error
0.0	0.92426	0.993641	6.9829×10^{-2}
0.1	1.12462	1.09814	2.4109×10^{-2}
0.2	1.20895	1.21364	3.8583×10^{-3}
0.3	1.34116	1.34127	8.5892×10^{-5}
0.4	1.4785	1.48234	2.5860×10^{-3}
0.5	1.6344	1.63824	2.3413×10^{-3}
0.6	1.80684	1.81053	2.0400×10^{-3}

Value of	Saulyev's Scheme	Exact Solution	Relative Error
0.7	1.9976	2.00095	1.6703×10^{-3}
0.8	2.20869	2.21139	1.2183×10^{-3}
0.9	2.44233	2.44396	6.6779×10^{-4}
1.0	2.71801	2.71801	0.0000

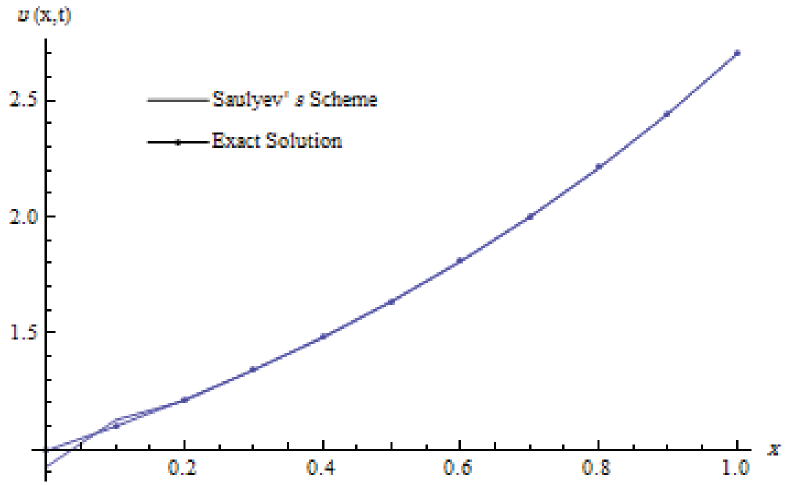

Figure (13): Saulyev's method and exact solution at ,$h = 0.1$, $k = 0.02$ and $t = 0.08$.

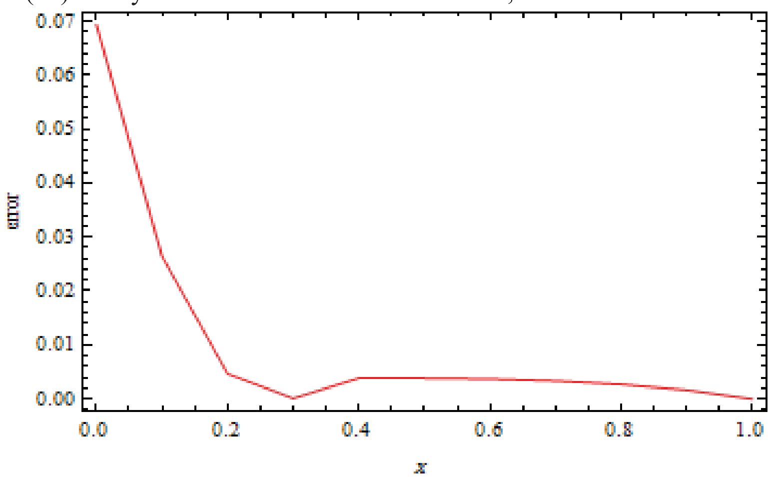

Figure (14): Error curve of Saulyev's method at $h = 0.1$, $k = 0.02$ and $t = 0.08$.

Obviously, the solutions in Tables (4) and (5) are closed to the exact solutions.

Although these methods are explicit but forward time central space is better than Saulyev's method as long as it is falling under the condition $r \leq 0.5$. In the other hand, Saulyev's method gave a good approximation of the exact solution at, $r = 2$, where $h = 0.1$, $k = 0.02$ and $t = 0.08$, , as shown in Table (6) and Figure (13). The reason is that, this scheme is unconditionally stable.

6. Conclusion

In this paper we have presented two numerical methods which are forward time central space (FTCS) and Saulyev's method for solving parabolic equation with nonlocal condition. Those methods are explicit and simple for implementation, due to the value of u_i^{j+1} can be updated independently of each other.

The entire solution is contained in two loops: an outer loop over all time steps, and an inner loop over all interior nodes. FTCS method is an applicable technique and approximates the exact solution for the giving problem very well, compared with the Saulyev's method, but the disadvantage of FTCS method is that h and k must be chosen to satisfy $r \leq 0.5$ to be stable. In contrary, Saulyev's method has no condition for stability.

Numerical experimental work showed us that, FTCS method is a preferable because the results are close to the exact solution, but behave badly as r becomes large than the condition limit, while Saulyev's method gives a good agreement for that limit. The error decreases with small value of k and increases as k made time error. Therefore, both of these methods are complementary. In particular, the choosing of these methods depend on the problem, if it requires that t and x satisfy the stability condition $r \leq 0.5$, then for this case we should use FTCS method. However, Saulyev's scheme is still useful for other problems.

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