Complete Analytic Vector Fields on Open Unit Ball in R^n

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• Abstract:

We shall give complete description of the analytic vector fields on an open unit Euclidean ball, $B = \{x: (\langle x,x \rangle) < 1\}$ in R^n with Vanish on unit sphere $S = \{x: \langle x,x \rangle = 1\}$ in R^n .

Keywords : Vector fields, Complete vector field, Analytic mapping, Analytic function, Cayley trans formation.

■ ملخص:

سوف نقدم في هذه الورقة البحثية وصف كامل لتركيبة المتجهات الحقلية على كرة الوحدة المفتوحة $B = \{x: (x . x) < 1\}$

في IRN ، $B \subset IR^N$ والتي تنتهي على ∂B

حيث $\partial B = \{x: (x . x) = 1\}$ في IRN حيث $\partial B \subset IR^N$.

Introduction

In our paper [1] we gave the complete structure of complete polynomial vector fields on open unit ball B: = $\{x: \langle x,x \rangle < 1\}$ in R^n ,

where $x = (x_1, x_2, \ldots, x_n)$ our result was the form :

$$P(x) = R(x) - \langle R(x), x \rangle x + (1 - \langle x,x \rangle)\, Q(x)$$

for some polynomial mappings R, $Q: R^n \to R^n$, In [2] we proved that if $F: R^N \to R$, is a polynomial and $P: R^N \to R$, be any polynomial such that $P(M) = 0$ and $M \subset R^N$ then there is a polynomial

$q: R^N \to R$, such that $P = q . F$ when $F(x) = Q_1(x) . Q_2(x) . \ldots . Q_N(x)$,

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and Q_i are linearly independent affine functions.

In [1] we proved that if $F: R^N \rightarrow R$, be a polynomial such that f(x) = 0 for $X \in S$ where S = {x:<x,x> = 1}. Then there exists a polynomial $Q: R^n \rightarrow R$, such that f(x) = (1- <x,x>) Q(x).

Our main Result.

Definition : Given any subset K in R^n , the set of n tuples and mapping $V: R^n \rightarrow R^n$, we say that V is complete victor field in K . If for every point $k_0 \in K$. There exists a curve $X: R \rightarrow K$ such that $X(0) = k_0$,

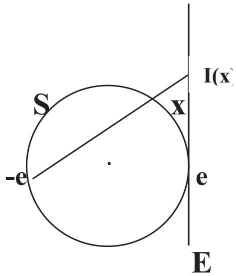
and $\frac{dx(t)}{dt} = V(x(t))$ for all $t \in R$.

CAYLEY TRANSFORM – [2]

Inversion:

$I : R^N \setminus \{-e\} \leftrightarrow R^N \setminus \{-e\}$.

$$I(x) = -e + 4\frac{x+e}{\|x+e\|^2}$$



Well-known : $I = I^{-1}$

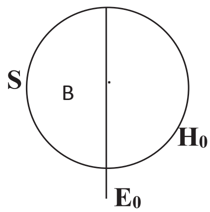
$I : S \setminus \{-e\} \leftrightarrow E$, where S is unit sphere and $E := \{x : x - e \perp e\}$

Cayley transform:

$$C(x) = I(x) - e = -2e + 4\frac{x+e}{\|x+e\|^2}$$

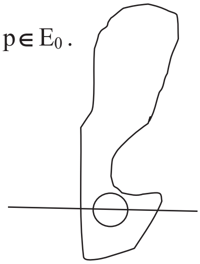
$C : S \setminus \{-e\} \leftrightarrow E_0 = E - e_0 = \{x : x \perp e\}$

$B = \{ x : \|x\| < 1\} \leftrightarrow H_0 = \{x : < x.e > \; > 0\}$



Theorem 1: Assume that $G \subset R^N$ is an open connected set such that $G \cap E_0 \neq \varnothing$ And let $\phi : G \leftrightarrow R$ be an analytic function such that $\phi(x) = 0$ for all $x \in G \cap E_0$. Then $\phi(x) = x_1 \Psi(x)$ for some analytic function $\Psi : G \leftrightarrow R$ where $x_1 = < x.e >$ and $x = (x_1, x_2, \dots, x_N) \in R^N$.

Proof: Let $E_0 = \{ p\in R^N \ : \ x_1(p) = 0 \}$, be a hyper-plane $\phi(p)=0$ for $p\in E_0$.



E0

$$\phi(p)=\sum_{k=1}^{\infty}\sum_{n_1+...+n_N=h} a_{n_1...n_N}\, x_1^{n_1}(p)...x_N^{n_N}(p)$$

$p \in E_0 \Rightarrow x_1(p)=0,\ x_1^{n_1}(p)...x_N^{n_N}(p)=0$, if $n_1 > 0$

$$0=\phi(p)=\sum_{k=1}^{\infty}\sum_{n_2+...+n_N=k} a_{0n_2...n_N}\, x_2^{n_2}(p)...x_N^{n_N}(p),$$ by assumption.

$p:=\xi_2 e_2+...+\xi_N e_N \in E_0$, $\xi_2,...,\xi_N \in R$, arbitrary.

$$0=\phi(p)=\sum_{n_2+...+n_N=k} a_{0n_2...n_N}\, \xi_2^{n_2}...\xi_N^{n_N}$$

$$a_{0\,n_2...n_N}=\frac{\partial^{n_2+...+n_N}\phi(\xi_2 e_2+...+\xi_N e_N)}{\partial x_2^{n_2}...\partial x_N^{n_N}}\,\frac{1}{n_2!\,...n_N!}=0.$$

$$a_{0\,n_2...n_N} = 0, \forall n_2,..,n_N.$$

$$\phi(p)=\sum_{k=1}^{\infty}\sum_{\substack{n_1+...+n_N=k\\ n_1>0}} a_{n_1n_2...n_N}\, x_1^{n_1}(p)...x_N^{n_N}(p)$$

$$= x_1(p)\sum_{k=1}^{\infty}\sum_{\substack{n_1+...+n_N=k\\ n_1>0}} a_{n_1n_2...n_N}\, x_1^{n_1-1}(p)...x_N^{n_N}(p)$$

$$= x_1(p)\ \psi(p).$$

$$\psi(p)=\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n_2+...+n_N=l} a_{n_1n_2...n_N}\, x_1^{m}(p)...x_N^{n_N}(p)$$

With $m=n_1-1$ and $l=k-1$. ■

Theorem 2 : Assume $u \subset R^N$ is an open connected set such that $-e \notin u$ and $S\cap u \neq$ and let $f:u\to R$ be an analytic function such that $f(x)= 0$ for all $x \in S\cap u$.

Then $f(x)= (\|x\|^2-1)\, g(x)$ for some analytic function $g:u\to R$.

Proof: Define $G:=C(u)$ and $\phi:=f \circ C^{-1}$ is terms of the Cayley transform C. Then $\phi(x)=0$ for all $x\in C(u\cap S) =C(u\cap[S \setminus <o>]) = C(u) \cap C(S \setminus <o>) = G\cap E_0 \neq\emptyset$.

Then by the Theorem 1 , we can write $\phi(x)=x_1\psi(x)$ with an analytic function $\psi:G\to R$. Since f = ϕ o C, we have ,

$$f(x) = [\,C(x)]_1\ \psi(C(x))\quad (x\in u).$$

Let us calculate $[\,C(x)\,]_1$:

$$[\,C(x)\,]_1 = \langle C(x),e\rangle = \left\langle -2e + 4\frac{x+e}{\|x+e\|^2}\ \middle|\ e\right\rangle$$

$$= -2+4\frac{x_1+1}{\|x+e\|^2} = -2+\frac{4x_1+4}{\|x\|^2+2\langle x|e\rangle+\|e\|^2}$$

$$= -2+\frac{4x_1+4}{\|x\|^2+2x_1+1} = \frac{1}{\|x\|^2+2x_1+1}\left[-2\|x\|^2-4x_1-2+4x_1+4\right]$$

$$= \frac{2-2\|x\|^2}{\|x+e\|^2}.$$

Therefore the function

$$g(x) := -2\frac{\psi(C(x))}{\|x+e\|^2}\qquad (x\in u).$$

Satisfy our requirements. ■

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